On weight-equitable partitions of graphs

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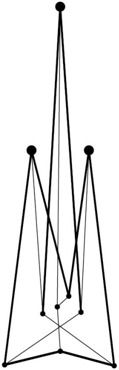
Outline

Introduction

Graph spectrum



$$\Longleftrightarrow \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

spectrum (eigenvalues): $\lambda_1 \geq \cdots \geq \lambda_n$

Graph spectrum

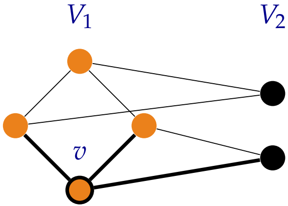
$\Longleftrightarrow$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

spectrum (eigenvalues): $\lambda_1 \geq \cdots \geq \lambda_n$

Equitable partitions

$$\mathcal{P} = \{V_1, V_2\}$$



$$B = (b_{ij}) = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}$$

$b_{11}(v) = 2, b_{12}(v) = 1$

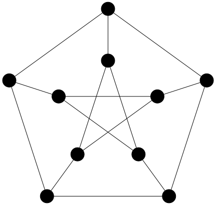
Equitable if b_{ij} only depends on i and j.

Representing partitions

$$\mathcal{P} = \{V_1, V_2, \ldots, V_m\}$$

$$S = \begin{bmatrix} | & & & \\ e_1 & & & \\ | & & & \\ & | & & \\ & e_2 & & \\ & | & & \\ & & \vdots & \\ & & & | \\ & & & e_m \\ & & & | \end{bmatrix}$$

Shrinking graphs while preserving (part) of the spectrum



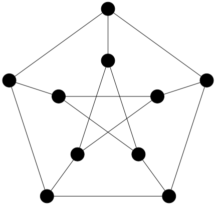
$\{3, 1^5, -2^4\}$

$\longrightarrow$

$$B = (b_{ij}) = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

$\{3, 1, -2\}$

Shrinking graphs while preserving (part) of the spectrum



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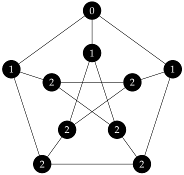
$\{3, 1^5, -2^4\}$ $\qquad$ $\{3, 1, -2\}$

Theorem (e.g. Cvetković, Doob, Sachs 1980)
Every eigenvalue of B is also an eigenvalue of $A(G)$.

Equitable partitions in algebraic combinatorics

Equitable partitions in algebraic combinatorics

- Naturally occur in graphs with rich algebraic structures:



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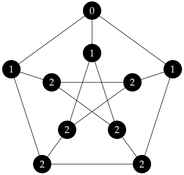
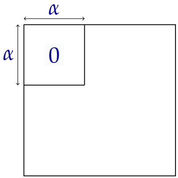


- Useful for proving eigenvalue bounds on graph parameters like the k-independence number (Cvetković 1972), (Haemers 1995), (A., Coutinho, Fiol 2019)



Extending equitable partitions

Equitable: every neighbor contributes to b_{ij} equally.

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What if we assign weights to the vertices?

Extending equitable partitions

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Use weights which ‘regularize’ the graph.

Perron eigenvector

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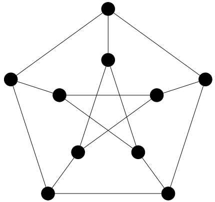


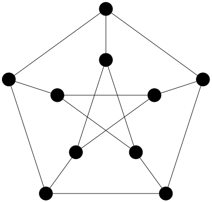
$\lambda_1 = 3$, $v = \mathbf{1}$

Perron eigenvector

Let G be a connected graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$.

Perron-Frobenius Theorem

- λ_1 is simple;
- The eigenvector ν of λ_1 has positive entries;
- All other eigenvectors have a negative entry.



$\lambda_1 = 3$, $\nu = \mathbf{1}$

We call ν the *Perron eigenvector*.

Weight partitions

$\mathcal{P} = \{V_1, V_2, \ldots, V_m\}$

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Weight quotient matrix $B^* = (b^*_{ij})$ with entries (*weight-intersection numbers*):

$$b^*_{ij}(u) := \frac{1}{\nu_u} \sum_{v \in G(u) \cap V_j} \nu_v \qquad u \in V_i$$

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Note that the sum of the weight-intersection numbers for all $1 \leq j \leq m$ gives the weight-degree of each vertex $u \in V_i$:

$$\sum_{j=1}^{m} b^*_{ij}(u) = \frac{1}{\nu_u} \sum_{v \in G(u)} \nu_v = \delta^*_u = \lambda_1$$

Weight-equitable partitions

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$$b^*_{ij}(u) := \frac{1}{\nu_u} \sum_{v \in G(u) \cap V_j} \nu_v \qquad \forall u \in V_i$$

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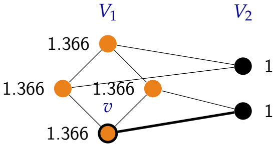


$b^*_{12}(v) = \frac{1}{\nu_v} \sum_{w \in G(v) \cap V_2} \nu_w = \frac{1}{1.366} \cdot 1 = 0.732$

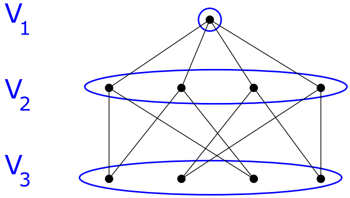
$B^* = (b^*_{ij}) = \begin{pmatrix} 2 & 0.732 \\ 2.732 & 0 \end{pmatrix}$

Weight-equitable if b^*_{ij} only depends on i and j.

Note: $\sum_j b^*_{ij} = \lambda_1$.

Example weight-equitable partition

$$\nu = (\ 2j \mid \sqrt{2}j \mid 1j\ )$$



$$b^*_{ij}(u) := \frac{1}{\nu_u} \sum_{v \in G(u) \cap V_j} \nu_v$$

$$b^*_{12}(1) = \tfrac{1}{2}(\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2})$$

$$b^*_{21}(2) = \tfrac{1}{\sqrt{2}}2$$

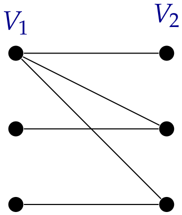
$$b^*_{21}(3) = \tfrac{1}{\sqrt{2}}2$$

$$b^*_{21}(4) = \tfrac{1}{\sqrt{2}}2$$

$$b^*_{21}(5) = \tfrac{1}{\sqrt{2}}2$$

...

Example weight-equitable partition but not equitable



$$\nu = (2.732, 1, 1, 1.414, 1.932, 1.932)$$

equitable $\quad \not\Leftarrow\!\!\Rightarrow \quad$ weight-equitable

Origin of weight-equitable partitions

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Ratio bound (Hoffman 1970)

If G is regular with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, then

$$\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}.$$

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Origin weight partitions

number $\chi(G)$ of G, is bounded below by $n/(\alpha(G))$. Thus upper bounds for $\alpha(G)$ give lower bounds for $\chi(G)$. For instance, if G is regular, Theorem 3.2 implies that $\chi(G) \geqslant 1 - \lambda_1/\lambda_n$. This bound, however, remains valid for nonregular graphs (but note that it does not follow from Theorem 3.3).

THEOREM 4.1.

(i) *If G is not the empty graph, then* $\chi(G) \geqslant 1 - (\lambda_1/\lambda_n)$.
(ii) *If $\lambda_2 > 0$, then* $\chi(G) \geqslant 1 - (\lambda_{n-\chi(G)+1}/\lambda_2)$.

Proof. Let $X_1, \dots, X_\chi$ $[\chi = \chi(G)]$ denote the color classes of G and let $u_1, \dots, u_n$ be an orthonormal set of eigenvectors of A (where u_i corresponds to λ_i). For $i = 1, \dots, \chi$, let s_i denote the restriction of u_1 to X_i, that is,

$$(s_i)_j = \begin{cases} (u_1)_j, & \text{if } j \in X_i, \\ 0, & \text{otherwise,} \end{cases}$$

and put $\tilde{S} = [s_1 \cdots s_\chi]$ (if some $s_i = 0$, we delete it from $\tilde{S}$ and proceed similarly) and $D = \tilde{S}^\mathsf{T}\tilde{S}$, $S = \tilde{S}D^{-1/2}$, and $B = S^\mathsf{T}AS$. Then B has zero diagonal (since each color class corresponds to a zero submatrix of A) and an eigenvalue λ_1 ($d = D^{1/2}\underline{1}$ is a λ_1-eigenvector of B). Moreover, interlacing Theorem 2.1 gives that the remaining eigenvalues of B are at least λ_n. Hence

$$0 = \operatorname{tr}(B) = \mu_1 + \cdots + \mu_\chi \geqslant \lambda_1 + (\chi - 1)\lambda_n,$$

which proves (i), since $\lambda_n < 0$. The proof of (ii) is similar, but a bit more

Origin weight-equitable partitions

Formally defined and used by
(Garriga, Fiol 1999)

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Theory of eigenvalue interlacing
extended (Fiol 1999)

Motivation

Why using weight-equitable partitions?

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Powerful tool used to extend several spectral bounds known for regular graphs also for **non-regular graphs**.

Applications of weight-equitable partitions

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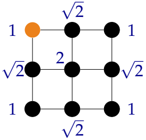
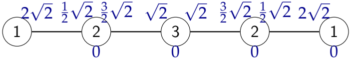
- (Fiol, Garriga, Yebra 1996) (Locally) pseudo-distance-regular graphs.





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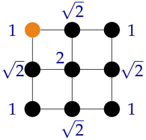

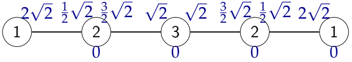

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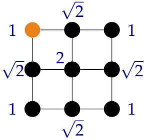

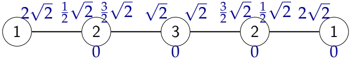

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- (Lee, Weng 2012) Spectral excess theorem for irregular graphs.

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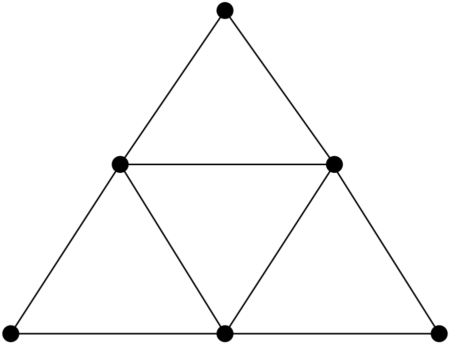
Equitable vs Weight-equitable partitions



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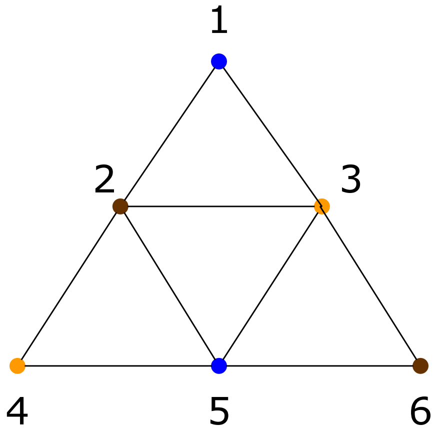

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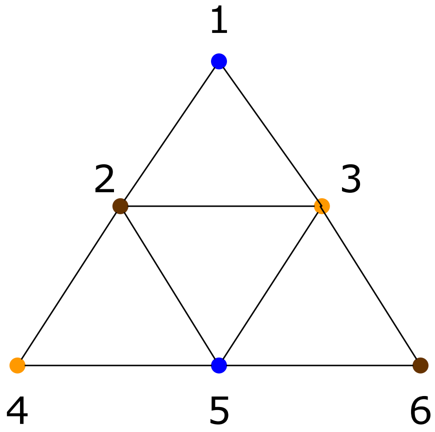

weight-equitable BUT NOT equitable

Equitable vs Weight-equitable partitions

$$\text{Equitable} \implies \text{Weight-equitable}$$

Equitable vs Weight-equitable partitions

Equitable $\Longrightarrow$ Weight-equitable

Converse not true!

Equitable $\not\Longleftarrow$ Weight-equitable

Relation between (weight-)equitable partitions

	graph class admitting ... partition with m cells	
number of cells m	equitable	weight-equitable
1	$\Longleftrightarrow$ regular	all
2	biregular	bipartite
n	all	all

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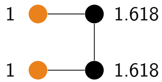
Proposition

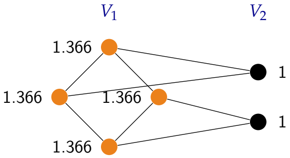
Weight-equitable and ν constant over all cells $\Leftrightarrow$ equitable



Spectral properties

Notation

Let $\rho : U \mapsto \sum_{u \in U} v_u e_u$.



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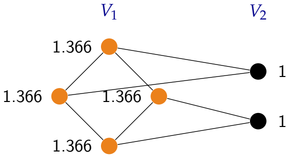
Let $\rho : U \mapsto \sum_{u \in U} \nu_u e_u$.



$$\bar{S}^* = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Normalized weight-characteristic matrix: $\bar{s}^*_{ui} = \begin{cases} \frac{\nu_u}{\|\rho(V_i)\|} & \text{if } u \in V_i, \\ 0 & \text{otherwise.} \end{cases}$

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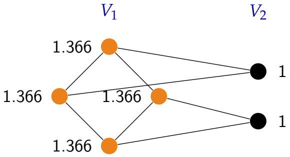

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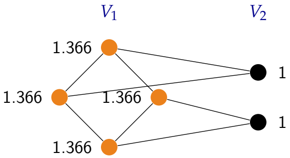

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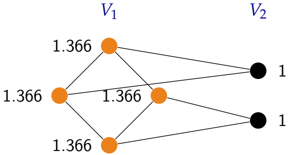

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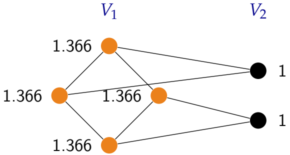

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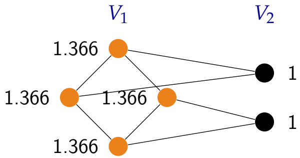
Notation

Let $\rho : U \mapsto \sum_{u \in U} v_u e_u$.



$$\bar{S}^* = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \bar{B}^* = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}$$

$$(\bar{S}^*)^\top \bar{S}^* = I \qquad \bar{B}^* = (\bar{S}^*)^\top A \bar{S}^*$$

Theorem

- $\bar{B}^*$ has largest eigenvalue λ_1;
- All eigenvalues of $\bar{B}^*$ are eigenvalues of G.

Motivation

It is often useful (why, in next section) to know whether a graph admits a weight-equitable partition :

Motivation

It is often useful (why, in next section) to know whether a graph admits a weight-equitable partition :

$\rightarrow$ Find characterizations and conditions.

Characterization I:

generalized double stochastic matrices and weight-regularity

Known characterizations

Theorem (Fiol 1999)
$AS^* = S^*B^* \iff \mathcal{P}$ weight-equitable partition

Double stochastic matrices

A matrix is *double stochastic* if it is nonnegative and each of its rows and each of its columns sums up to one.

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Note: $\Omega(A)$ is a convex polytope since it consists of all matrices X such that

$$XA = AX, \quad X\mathbf{1} = \mathbf{1}X = 1, \quad X \geq 0.$$

Double stochastic matrices and equitable partitions

Lemma (Godsil 1997)

Let A be the adjacency matrix of a graph G, and let $\mathcal{P}$ be a partition of the vertex set with normalized characteristic matrix S. Then, $\mathcal{P}$ is equitable if and only if A and $SS^\top$ commute.

Double stochastic matrices and equitable partitions

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Question: Can we extend this result to weight-equitable partitions?

Generalized double stochastic matrices

A matrix is *generalized double stochastic* if it is nonnegative and each of its rows and each of its columns sums up to the same constant.

Generalized double stochastic matrices

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Note: $\Omega^*(A)$ is also a convex polytope since it consists of all matrices X such that

$$XA = AX, \quad X\mathbf{1} = \mathbf{1}X, \quad X \geq 0.$$

Generalized double stochastic matrices and weight-equitable partitions

Lemma (A. 2019)
Let A be the adjacency matrix of a graph G, and let $\mathcal{P}$ be a weight partition of the vertex set with normalized weight-characteristic matrix $\overline{S}^*$. Then, $\mathcal{P}$ is weight-equitable if and only if A and $\overline{S}^*\overline{S}^{*\top}$ commute.

Corollary (A. 2019)
Let $\mathcal{P}$ be a weight partition of the vertex set of a graph with normalized weight-characteristic matrix $\overline{S}^*$. Then $\mathcal{P}$ is weight-equitable if and only if $\overline{S}^*\overline{S}^{*\top} \in \Omega^*(A)$.

Characterization II:

Fractional automorphisms and weight-regularity

Fractional automorphisms

A adjacency matrix of a graph

Fractional automorphisms

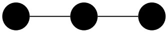
A adjacency matrix of a graph

Graph automorphism:

Permutation matrix P
s.t. $PA = AP$

Fractional automorphism:

Doubly stochastic matrix X
s.t. $XA = AX$



$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

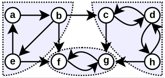
$$X = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

Fractional automorphisms

Let $X = (x_{ij})$ be doubly stochastic and define the directed graph G_A with adjacency matrix

$$A = (a_{ij}) = \begin{cases} 1 & \text{if } x_{ij} \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and let $\mathcal{P}_X$ be the partition of $[n]$ into the strongly connected components of G_A.



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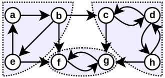

Proposition (A., Hojny, Zeijlemaker 2022)
If X commutes with $A(G)$, then $\mathcal{P}_X$ is weight-equitable.

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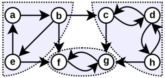

Proposition (A., Hojny, Zeijlemaker 2022)
If X commutes with $A(G)$, then $\mathcal{P}_X$ is weight-equitable.

Unfortunately no hope for an iff result ...

Fractional automorphisms

Proposition (A., Hojny, Zeijlemaker 2022)

Given a partition $\mathcal{P}$, let $X_{\mathcal{P}}$ be a matrix with entries

$$x_{vw} = \begin{cases} \frac{\nu_v \nu_w}{\|\rho(P)\|^2} & \text{if } v, w \in P \text{ for some } P \in \mathcal{P}, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{P}$ is a weight-equitable partition, then $X_{\mathcal{P}} A = A X_{\mathcal{P}}$.

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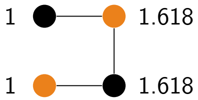

$$X_{\mathcal{P}} = \begin{pmatrix} 0.276 & 0 & 0.447 & 0 \\ 0 & 0.724 & 0 & 0.447 \\ 0.447 & 0 & 0.724 & 0 \\ 0 & 0.447 & 0 & 0.276 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$X_{\mathcal{P}}$ not a double stochastic, but quite symmetric ...

Characterization III:

Hoffman-type polynomial and weight-regularity

Related results

(Hoffman 1963)
Characterization of regular graphs in terms of the Hoffman polynomial.

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(A., Dalfó, Fiol 2013)
Hoffman's-like characterization for biregular graphs.

Related results

Theorem (A., Dalfó, Fiol 2013)
A bipartite graph $G = (V_1 \cup V_2, E)$, with $n = n_1 + n_2$ vertices in (δ_1, δ_2)-biregular if and only if the odd part of its preHoffman polynomial satisfies

$$H_1(A) = \alpha \begin{pmatrix} \boldsymbol{O} & J \\ J & \boldsymbol{O} \end{pmatrix}$$

with $\alpha = \frac{n_1+n+2}{2\sqrt{n_1 n_2}} = \frac{\delta_1+\delta_2}{2\sqrt{\delta_1\delta_2}}$.

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Question: Can we find a Hoffman-like polynomial to characterize weight-regularity?

Polynomials and weight-regularity

Theorem (A. 2019)

Let G be a connected graph with a partition of its vertices into m sets, $\mathcal{P} = \{V_1, \ldots, V_m\}$, such that $n = n_1 + \cdots + n_m$ and such that the map on V, denoted by $\boldsymbol{\rho} : u \to \nu_u$, is constant over each V_k. Then there exists a polynomial $H \in \mathbb{R}_d[x]$ such that

$$H(A) = \begin{pmatrix} b^*_{11}J & b^*_{12}J & \cdots & b^*_{1m}J \\ b^*_{21}J & b^*_{22}J & \cdots & b^*_{2m}J \\ \vdots & & \ddots & \\ b^*_{m1}J & b^*_{m2}J & \cdots & b^*_{mm}J \end{pmatrix}$$

if and only if $\mathcal{P}$ is a weight-equitable partition of G.

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Weight-equitable partitions maintain the structure of the Perron eigenvector $\boldsymbol{\nu}$.

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if and only if $\mathcal{P}$ is a weight-equitable partition of G.

As a corollary, for a regular graph $\nu = 1$: (Hoffman 1963)

An application to graph theory: improvement of Hoffman's bound

Hoffman's ratio bound

Theorem (Hoffman 1970)

If G has at least one edge, then

$$\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}.$$

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If G has at least one edge, then

$$\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}.$$

When equality holds we call the coloring a *Hoffman coloring*.

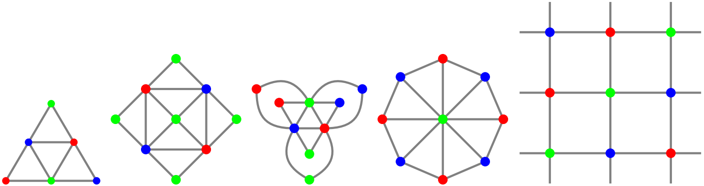
Examples of Hoffman colorable graphs

Trivially Hoffman colorable graphs:

- Bipartite graphs;
- Regular complete multipartite graphs (e.g. $K_{3,3,3}$), including complete graphs.

BUT not many non-trivial infinite families of Hoffman colorable graphs are known!

Some irregular Hoffman colorable graphs:



Hoffman colorings in the literature

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- (A., Bosma, Van Veluw 2025) Structural properties of Hoffman colorings of irregular graphs.

Motivation to study Hoffman colorings

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$$h(G) \leq \chi_v(G) \leq \chi_{sv}(G) \leq \chi_q(G) \leq \chi(G)$$

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$h(G) = 1 - \frac{\lambda_1}{\lambda_n} \leq \chi(G)$

- For Hoffman colorable graphs, h is optimal,
- Improving Hoffman's bound,
- Sandwiching: if $h(G) = \chi(G)$ then

$$h(G)=\chi_v(G)=\chi_{sv}(G)=\chi_q(G)=\chi(G)$$

Exact values of chromatic parameters for free, even some that are not known to be computable like $\chi_q(G)$!

Improving Hoffman's bound

Theorem (A. 2019)

If G has chromatic number $\chi(G)$ and a Hoffman coloring, then the partition defined by the color classes is weight-equitable.

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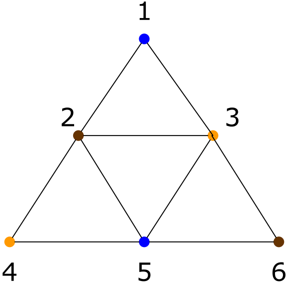

$$\mathcal{P} = \{V_1, V_2, V_3\} = \{\{2,6\}, \{1,5\}, \{3,4\}\}$$

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What can we do with this theorem?

Corollary (A. 2019)

If G has at least one edge and the vertex partition defined by the χ color classes is not weight-equitable, then

$$\chi(G) \geq 2 - \frac{\lambda_1}{\lambda_n}.$$

Improving Hoffman's bound

Theorem (A. 2019)

If G has chromatic number $\chi(G)$ and a Hoffman coloring, then the partition defined by the color classes is weight-equitable.

What can we do with this theorem?

If G does not have a weight-regular partition
$\Longrightarrow G$ cannot have a Hoffman coloring
$\Longrightarrow$ useful for finding families of non-regular Hoffman colorable graphs (A., Bosma, Van Veluw 2025)

Computing weight-equitable partitions

Complexity equitable partitions

https://mathoverflow.net/questions/96858/complexity-of-equitable-partitions

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Complexity of equitable partitions

Asked 11 years, 5 months ago   Modified 6 years, 5 months ago   Viewed 2k times

18

We are talking about undirected simple graphs and *partitions* of their vertex sets into disjoint non-empty *cells*. Such a partition is *equitable* if for any two vertices v, w in the same cell, and any cell C, it holds that v, w have the same number of neighbours in C. The *trivial* partition (with only one vertex per cell) is always equitable.

Given any partition π, there is a unique coarsest equitable partition $\overline{\pi}$ finer than π. (The concepts *finer* and *coarser* include equality). This is a very old result, as also are polynomial-time algorithms for computing $\overline{\pi}$ from π.

Another fact is that it is NP-complete to determine if a graph has an equitable partition with every cell of size 2. (This follows from Lubiw, SIAM J Comput 10, 1981, 11–21 on noting that such a partition corresponds to a fixed-point-free automorphism of order 2.)

My question is: **what else?** Are any other complexity results known? In particular:

1. What is the complexity of: Given a regular graph, does it have any non-trivial equitable partition other than the partition with just one cell?
2. What is the complexity of: Given a regular graph, does it have an equitable partition with exactly two cells?
3. What is the complexity of: Given a graph and two vertices v, w, is there a non-trivial equitable partition which has v, w in different cells?
4. Is there any problem on equitable partitions with complexity equal to graph isomorphism?

graph-theory   computational-complexity

Share Cite Improve this question   edited May 14, 2012 at 9:09   asked May 13, 2012 at 23:34

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Applications and coarseness

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Depending on the application, different levels of coarseness are required:

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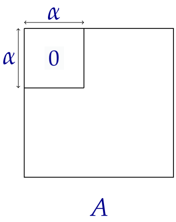
- Bounds on α: two cells;
- Pseudo-distance-regular graphs: $\#$cells $=$ diameter $+ 1$.



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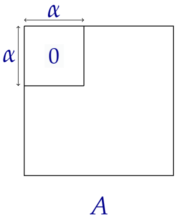
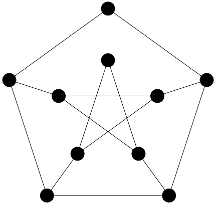


In general, coarser means more shrinkage.

Shrinking graphs while preserving (part) of the spectrum



$\{3, 1^5, -2^4\}$

$$\longrightarrow \quad B = (b_{ij}) = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

$\{3, 1, -2\}$

Our focus

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Coarse(st) weight-equitable partitions

- Largest size reduction;
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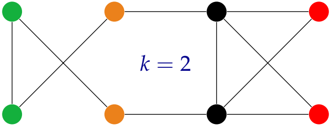
k-homogeneous weight-equitable partitions



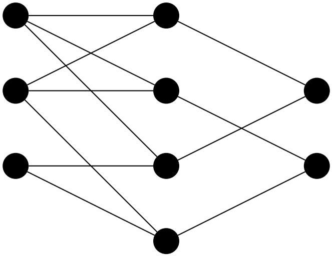
- Finding general complexity results is hard, so start with a regular case.

Computational results for equitable partitions: coarsest

Theorem (Bastert 1999)

The coarsest equitable partition of a graph can be found in polynomial time.

Color splitting:



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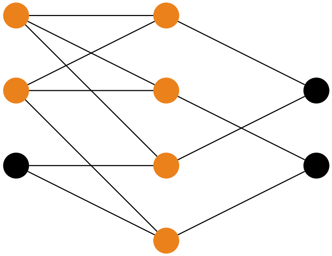

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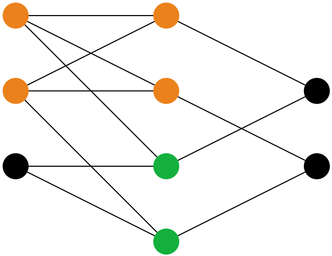

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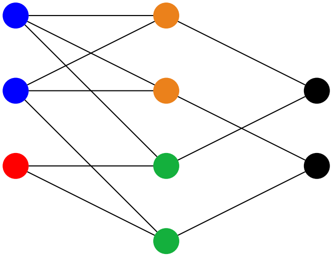

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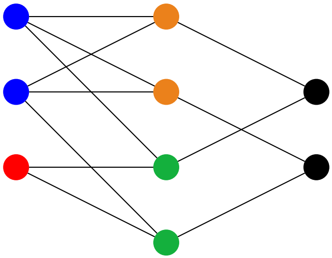

Based on algorithm for minimising finite automata (Hopcroft 1971).

Can be computed in $O(m \log n)$ time (Cardon, Crochemore 1982).

Extension to weight-equitable partitions?

For finding the coarsest equitable partition, the principle of the coloring algorithm seems to work. . .

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..but the starting partition is already weight-equitable!

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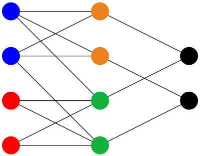
..but the starting partition is already weight-equitable!

	graph class admitting . . . partition with m cells	
number of cells m	equitable	weight-equitable
1	$\Longleftrightarrow$ regular	**all**
2	biregular	bipartite
n	all	all

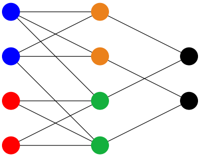
Computational results for equitable partitions: 2-homogeneous

Lemma
2-homogeneous equitable partition $\Leftrightarrow$ the graph has an automorphism being an *involution without fixed points* (autom of the graph where every vertex is in a pair).



Computational results for equitable partitions: 2-homogeneous

Lemma
2-homogeneous equitable partition $\Leftrightarrow$ the graph has an automorphism being an *involution without fixed points* (autom of the graph where every vertex is in a pair).



(Lubiw 1981)
Deciding whether a given graph has a fixed-point-free automorphism of order two is NP-complete.

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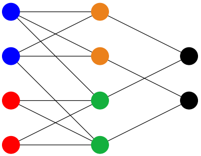

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Corollary

Finding 2-homogeneous equitable partitions is NP-complete.

Overview computational results equitable partitions

Coarsest:

Theorem (Bastert 1999)

The coarsest equitable partition of a graph can be found in polynomial time.

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Finding 2-homogeneous equitable partitions is NP-complete.

Extension to weight-equitable partitions?

2-homogeneous equitable partition NP-completeness
Cannot extend proof unless the partition also happens to be equitable.

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Question: When does this happen?

Maybe a polynomial algorithm for some graph class?

We just saw that in general, we cannot decide in polynomial time whether a graph admits an equitable partition with cells of size two.

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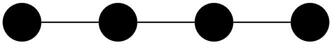
Question: Are there graph classes for which we can obtain an efficient algorithm to compute such 2-homogeneous equitable partitions?

A small example



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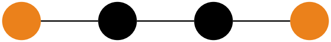
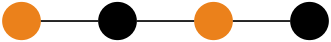


Equitable
(and hence weight-equitable)

A small example

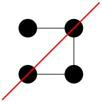


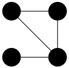
Not equitable
(but weight-equitable)

Graphs without P_4?

Cographs

Graphs without induced P_4





Cographs

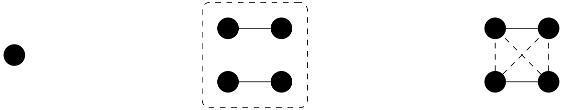
Graphs without induced P_4

(Corneil, Lerchs, Burlingham 1981)

(i) K_1 is a cograph

(ii) If $G_1, \ldots, G_k$ cographs, then $G_1 \cup \cdots \cup G_k$ cograph

(iii) The join of cographs is a cograph



(i) (ii) (iii)

Cographs

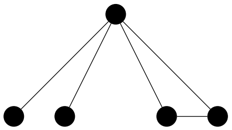
Graphs without <u>induced</u> P_4

Proposition (A., Hojny, Zeijlemaker 2022)
In cographs, all 2-homogeneous weight-equitable partitions are equitable.

Goal: devise algorithm to find 2-homogeneous weight-equitable partitions of cographs.

Representing cographs with trees

Cotree: Represent union with 0, join with 1



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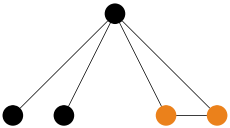

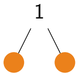

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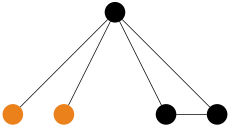

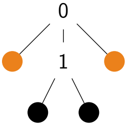

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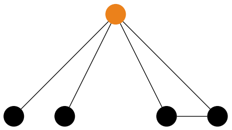

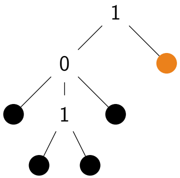

Note that leaves of the cotree are the vertices in the cograph.

(Corneil, Lerchs, Burlingham 1981)
If 0 and 1 alternate, this tree is unique.

Representing cographs with trees

Lemma (A., Hojny, Zeijlemaker 2022)
Automorphism ϕ of cograph G $\Leftrightarrow$ automorphism of the cotree which:

- acts as the original ϕ on the leaves,
- respects the $0/1$-labeling.

Representing cographs with trees

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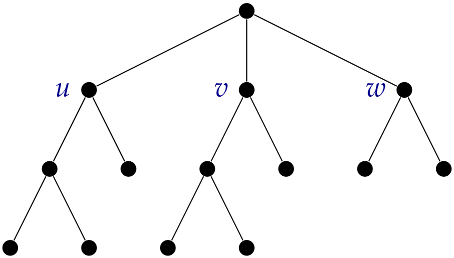
- respects the $0/1$-labeling.

This allows us to translate the problem of finding 2-homogeneous partitions to the problem of finding automorphisms in a tree ...

Computing 2-homogeneous equitable partitions: intuition

2-homogeneous equitable partition
$\overset{\text{Lubiw}}{\Longleftrightarrow}$ involution of the graph without fixed points (automorphisms of order 2 in the graph)
$\overset{\text{Lemma}}{\Longleftrightarrow}$ automorphism of cotree which is a fixed-point-free involution on the leaves.



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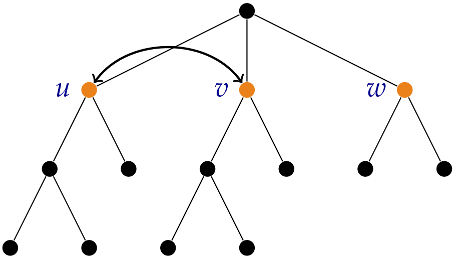

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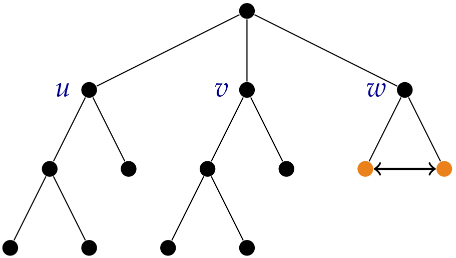

Computing 2-homogeneous equitable partitions: algorithm

Input : Labeled (co)tree T, root vertex r

for *each child of* r *with distinct subtree* **do**
- **if** *an odd number of children have the same subtree* **then**
  - **if** *the child is a leaf* **then**
    - **return** false
  - **else**
    - recurse

return true

Computing 2-homogeneous equitable partitions: algorithm

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for *each child of* r *with distinct subtree* **do**
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return true

Edmonds' algorithm

Computing 2-homogeneous equitable partitions: algorithm

Edmonds' algorithm (Busacker, Saaty 1965)
Algorithm for detecting isomorphic subtrees.

(Colbourn and Booth 1981)
Linear time extension of Edmonds' algorithm.

Theorem (A., Hojny, Zeijlemaker 2022)
Let G be a cograph. The problem of deciding whether G admits a (weight-) equitable partition with $\frac{n}{2}$ cells of size 2 can be solved in $O(n^2)$ time.

Closing remarks

Open problems: algebraic flavour

- New characterizations of weight-equitable partitions.
- Find new applications of weight(-equitable) partitions.

Open problems: algorithmic flavour

- Complexity of (weight-)equitable partitions?

https://mathoverflow.net/questions/96858/complexity-of-equitable-partitions

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Complexity of equitable partitions

Asked 11 years, 5 months ago Modified 6 years, 5 months ago Viewed 2k times

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We are talking about undirected simple graphs and *partitions* of their vertex sets into disjoint non-empty *cells*. Such a partition is *equitable* if for any two vertices v, w in the same cell, and any cell C, it holds that v, w have the same number of neighbours in C. The *trivial* partition (with only one vertex per cell) is always equitable.

Given any partition π, there is a unique coarsest equitable partition $\bar{\pi}$ finer than π. (The concepts *finer* and *coarser* include equality). This is a very old result, as also are polynomial-time algorithms for computing $\bar{\pi}$ from π.

Another fact is that it is NP-complete to determine if a graph has an equitable partition with every cell of size 2. (This follows from Lubiw, SIAM J Comput 10, 1981, 11–21 on noting that such a partition corresponds to a fixed-point-free automorphism of order 2.)

My question is: **what else?** Are any other complexity results known? In particular:

1. What is the complexity of: Given a regular graph, does it have any non-trivial equitable partition other than the partition with just one cell?
2. What is the complexity of: Given a regular graph, does it have an equitable partition with exactly two cells?
3. What is the complexity of: Given a graph and two vertices v, w, is there a non-trivial equitable partition which has v, w in different cells?
4. Is there any problem on equitable partitions with complexity equal to graph isomorphism?

graph-theory computational-complexity

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edited May 14, 2012 at 9:09

asked May 13, 2012 at 23:34
Brendan McKay

Thank you for your attention!

a.abiad.monge@tue.nl

Further reading:

A. Abiad
A characterization and an application of weight-regular partitions of graphs
Linear Algebra and Appl. 569 (2019).

A. Abiad, C. Hojny, S. Zeijlemaker.
Characterizing and computing weight-equitable partitions of graphs
Linear Algebra and Appl. 645 (2022).